

VANISHING THEOREMS ON COVERING MANIFOLDS

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ABSTRACT. Let M be an oriented even-dimensional Riemannian manifold on which a discrete group Γ of orientation-preserving isometries acts freely, so that the quotient $X = M/\Gamma$ is compact. We prove a vanishing theorem for a half-kernel of a Γ -invariant Dirac operator on a Γ -equivariant Clifford module over M , twisted by a sufficiently large power of a Γ -equivariant line bundle, whose curvature is non-degenerate at any point of M . This generalizes our previous vanishing theorems for Dirac operators on a compact manifold.

In particular, if M is an almost complex manifold we prove a vanishing theorem for the half-kernel of a spin^c Dirac operator, twisted by a line bundle with curvature of a mixed sign. In this case we also relax the assumption of non-degeneracy of the curvature. When M is a complex manifold our results imply analogues of Kodaira and Andreotti-Grauert vanishing theorems for covering manifolds.

As another application, we show that semiclassically the spin^c quantization of an almost complex covering manifold gives an “honest” Hilbert space. This generalizes a result of Borthwick and Uribe, who considered quantization of compact manifolds.

Application of our results to homogeneous manifolds of a real semisimple Lie group leads to new proofs of Griffiths-Schmidt and Atiyah-Schmidt vanishing theorems.

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This research was partially supported by grant No. 96-00210/1 from the United States-Israel Binational Science Foundation (BSF).

1. INTRODUCTION

One of the most fundamental results of the geometry of *compact* complex manifolds is the Kodaira vanishing theorem for the cohomology of the sheaf of sections of a holomorphic vector bundle twisted by a large power of a positive line bundle. Andreotti and Grauert [1] generalized this result to the case when the line bundle is not necessarily positive (but satisfies some non-degeneracy conditions, cf. Section 3).

Both the Kodaira and the Andreotti-Grauert theorems are equivalent to a vanishing of the kernel of the restriction of the Dolbeault-Dirac operator $\sqrt{2}(\bar{\partial} + \bar{\partial}^*)$ to the space of differential forms of certain degree. In [8], the author obtained a generalization of these results to abstract Dirac operators twisted by a large power of a line bundle (see also [7] where an analogue of the Kodaira vanishing theorem for spin^c Dirac operator on an almost Kähler manifold is proven).

In this paper we show that suitable generalizations of all the preceding vanishing theorems remain true if the base manifold is not compact but is an infinite normal covering of a compact manifold. The obtained results are very convenient for the study of homogeneous vector bundles over homogeneous spaces of real semisimple Lie groups. In particular, we obtain new proofs of certain results of Griffiths and Schmid [15] and Atiyah and Schmid [2], cf. Section 5.

As another application, we prove that *semiclassically the spin^c -quantization of an almost complex covering manifold gives an honest Hilbert space* and not just a virtual one, cf. Subsection 4.6.

We now give a brief review of the main results of the paper.

1.1. The L^2 vanishing theorem for the half-kernel of a Dirac operator. Suppose M is an oriented even-dimensional Riemannian manifold on which a discrete group Γ of orientation-preserving isometries acts freely, so that the quotient M/Γ is compact. Let $C(M)$ denote the Clifford bundle of M , i.e., a vector bundle whose fiber at any point is isomorphic to the Clifford algebra of the cotangent space. Let \mathcal{E} be a Γ -equivariant self-adjoint Clifford module over M , i.e., a Γ -equivariant vector bundle over M endowed with a Γ -invariant Hermitian structure and a fiberwise self-adjoint action of $C(M)$. Then (cf. Subsection 2.2) \mathcal{E} possesses a natural grading $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$. Let \mathcal{L} be a Γ -equivariant Hermitian line bundle endowed with a Γ -invariant Hermitian connection $\nabla^{\mathcal{L}}$. These data define (cf. Subsection 2.3) a Γ -invariant self-adjoint Dirac operator D_k acting on the space $L^2(M, \mathcal{E} \otimes \mathcal{L}^k)$ of square integrable sections of $\mathcal{E} \otimes \mathcal{L}^k$. The curvature $F^{\mathcal{L}}$ of $\nabla^{\mathcal{L}}$ is an imaginary valued 2-form on M . If it is non-degenerate at all points of M , then $iF^{\mathcal{L}}$ is a symplectic form on M , and, hence, defines an orientation of M . Our first result (Theorem 2.6) states that *the restriction of the kernel of D_k to $L^2(M, \mathcal{E}^- \otimes \mathcal{L}^k)$ (resp. to $L^2(M, \mathcal{E}^+ \otimes \mathcal{L}^k)$) vanishes for large k if this orientation coincides with (resp. is opposite to) the given orientation of M .*

1.2. The L^2 Andreotti-Grauert theorem. Suppose that a discrete group Γ acts holomorphically and freely of a complex manifold M , so that the quotient M/Γ is compact. Let \mathcal{W} be a holomorphic Γ -equivariant vector bundle over M and let \mathcal{L} be a holomorphic Γ -equivariant line bundle over M . Assume that \mathcal{L} carries a Γ -invariant Hermitian metric whose curvature form has at least q negative and at least p positive eigenvalues at any point $x \in M$. Then (Theorem 3.5),

the L^2 -cohomology $L^2 H^{0,j}(M, \mathcal{W} \otimes \mathcal{L}^k)$ of M with coefficients in the tensor product $\mathcal{W} \otimes \mathcal{L}^k$ vanishes for $j \neq q, q+1, \dots, n-p$ and $k \gg 0$.

In particular, if \mathcal{L} is a positive bundle, then $L^2 H^{0,j}(M, \mathcal{W} \otimes \mathcal{L}^k) = 0$, for $j \neq 0$ and $k \gg 0$. This is an L^2 -analogue of the Kodaira vanishing theorem.

1.3. A generalization to almost complex manifolds. If, in the conditions of the previous subsection, M is a Kähler manifold, then the L^2 cohomology $L^2 H^{0,*}(M, \mathcal{W} \otimes \mathcal{L}^k)$ is isomorphic to the kernel of the Dolbeault-Dirac operator $D_k : L^2 \mathcal{A}^{0,*}(M, \mathcal{W} \otimes \mathcal{L}^k) \rightarrow L^2 \mathcal{A}^{0,*}(M, \mathcal{W} \otimes \mathcal{L}^k)$, where $L^2 \mathcal{A}^{0,*}(M, \mathcal{W} \otimes \mathcal{L}^k)$ denotes the space of square integrable differential forms of type $(0,*)$ on M with coefficients in $\mathcal{W} \otimes \mathcal{L}^k$. This suggest a generalization of the L^2 Andreotti-Grauert theorem to the case when M is only an almost complex manifold.

Assume that \mathcal{L} possess a Hermitian connection whose curvature is a $(1,1)$ form on M which has at least q negative and at least p positive eigenvalues at any point $x \in M$. In this situation a Dirac operator $D_k : L^2 \mathcal{A}^{0,*}(M, \mathcal{W} \otimes \mathcal{L}^k) \rightarrow L^2 \mathcal{A}^{0,*}(M, \mathcal{W} \otimes \mathcal{L}^k)$ is defined, cf. Subsection 4.3. If the almost complex structure on M is not integrable, one can not hope that the kernel of D_k belongs to $\bigoplus_{j=q}^{n-p} L^2 \mathcal{A}^{0,j}(M, \mathcal{W} \otimes \mathcal{L}^k)$. However, we show in Theorem 4.4, that *for any $k \gg 0$ and any $\alpha \in \text{Ker } D_k$, “most of the norm” of α is concentrated in $\bigoplus_{j=q}^{n-p} L^2 \mathcal{A}^{0,j}(M, \mathcal{W} \otimes \mathcal{L}^k)$* . In particular, *if the curvature of \mathcal{L} is non-degenerate and has exactly q negative eigenvalues at any point of M , then “most of the norm” of $\alpha \in \text{Ker } D_k$ is concentrated in $L^2 \mathcal{A}^{0,q}(M, \mathcal{W} \otimes \mathcal{L}^k)$, and, depending on the parity of q , the restriction of the kernel of D_k either to $L^2 \mathcal{A}^{0,odd}(M, \mathcal{W} \otimes \mathcal{L}^k)$ or to $L^2 \mathcal{A}^{0,even}(M, \mathcal{W} \otimes \mathcal{L}^k)$ vanishes.*

In Subsection 4.6, we discuss applications of the above result to geometric quantization of covering manifolds. In this way we obtain L^2 analogues of results of Borthwick and Uribe [7].

Contents. The paper is organized as follows:

In Sections 2–4, we state our vanishing theorems for covering manifold.

In Section 5, we discuss applications of our results to representation theory of real semisimple Lie groups.

In Section 6, we present the proof of Theorem 2.6 (the vanishing theorem for the kernel of a Dirac operator). The proof is based on two statements (Propositions 6.3 and 6.4) which are proven in the later sections.

In Section 7, we prove an estimate on the Dirac operator on an almost complex manifold (Proposition 4.8) and use it to prove Theorem 4.4 (our analogue of the Andreotti-Grauert vanishing theorem for almost complex manifolds). The proof is based on Propositions 6.4 and 7.1 which are proven in later sections.

In Section 8, we prove the L^2 Andreotti-Grauert theorem (Theorem 3.5).

In Section 9, we use the Lichnerowicz formula to prove Propositions 6.3, 7.1 and 7.4. These results establish the connection between the Dirac operator and the rough Laplacian. They are used in the proofs of Theorems 2.6, 4.4 and 3.5.

Finally, in Section 10, we prove Proposition 6.4 (the estimate on the rough Laplacian).

Acknowledgments. I would like to thank Yael Karshon for valuable discussions.

2. L^2 VANISHING THEOREM FOR THE HALF-KERNEL OF A DIRAC OPERATOR

In this section we formulate one of the main results of the paper: the L^2 vanishing theorem for the half-kernel of a Dirac operator (cf. Theorem 2.6).

The section is organized as follows: in Subsections 2.1–2.3 we recall some basic facts about Clifford modules and Dirac operators. When possible we follow the notations of [5]. In Subsection 2.4 we discuss some properties of Dirac operator on covering manifolds. Finally, in Subsection 2.5 we formulate our main result.

2.1. Clifford Modules. Suppose M is an oriented even-dimensional Riemannian manifold and let $C(M)$ denote the Clifford bundle of M (cf. [5, §3.3]), i.e., a vector bundle whose fiber at any point $x \in M$ is isomorphic to the Clifford algebra $C(T_x^*M)$ of the cotangent space.

A *Clifford module* on M is a complex vector bundle \mathcal{E} on M endowed with an action of the bundle $C(M)$. We write this action as

$$(a, s) \mapsto c(a)s, \quad \text{where } a \in \Gamma(M, C(M)), s \in \Gamma(M, \mathcal{E}).$$

A Clifford module \mathcal{E} is called *self-adjoint* if it is endowed with a Hermitian metric such that the operator $c(v) : \mathcal{E}_x \rightarrow \mathcal{E}_x$ is skew-adjoint, for any $x \in M$ and any $v \in T_x^*M$.

A connection $\nabla^{\mathcal{E}}$ on a Clifford module \mathcal{E} is called a *Clifford connection* if

$$[\nabla_X^{\mathcal{E}}, c(a)] = c(\nabla_X a), \quad \text{for any } a \in \Gamma(M, C(M)), X \in \Gamma(M, TM).$$

In this formula, ∇_X is the Levi-Civita covariant derivative on $C(M)$ associated with the Riemannian metric on M .

Suppose \mathcal{E} is a Clifford module and \mathcal{W} is a vector bundle over M . The *twisted Clifford module obtained from \mathcal{E} by twisting with \mathcal{W}* is the bundle $\mathcal{E} \otimes \mathcal{W}$ with Clifford action $c(a) \otimes 1$. Note that the twisted Clifford module $\mathcal{E} \otimes \mathcal{W}$ is self-adjoint if and only if so is \mathcal{E} .

Let $\nabla^{\mathcal{W}}$ be a connection on \mathcal{W} and let $\nabla^{\mathcal{E}}$ be a Clifford connection on \mathcal{E} . Then the *product connection*

$$\nabla^{\mathcal{E} \otimes \mathcal{W}} = \nabla^{\mathcal{E}} \otimes 1 + 1 \otimes \nabla^{\mathcal{W}} \tag{2.1}$$

is a Clifford connection on $\mathcal{E} \otimes \mathcal{W}$.

2.2. The chirality operator. The natural grading. Fix $x \in M$ and let e_1, \dots, e_{2n} be an oriented orthonormal basis of T_x^*M . Consider the element

$$\gamma = i^n e_1 \cdots e_{2n} \in C(T_x^*M) \otimes \mathbb{C}, \tag{2.2}$$

called the *chirality operator*. It is independent of the choice of the basis, anti-commutes with any $v \in T_x^*M \subset C(T_x^*M)$, and satisfies $\gamma^2 = -1$, cf. [5, §3.2]. We also denote by γ the section of $C(M)$ whose restriction to each fiber is equal to the chirality operator.

The *natural grading* on a Clifford module \mathcal{E} is defined by the formula

$$\mathcal{E}^{\pm} = \{v \in \mathcal{E} : c(\gamma)v = \pm v\}. \tag{2.3}$$

Note that this grading is preserved by any Clifford connection on \mathcal{E} . Also, if \mathcal{E} is a self-adjoint Clifford module (cf. Subsection 2.1), then the chirality operator $c(\gamma) : \mathcal{E} \rightarrow \mathcal{E}$ is self-adjoint.

Hence, the subbundles \mathcal{E}^\pm are orthogonal with respect to the Hermitian metric on \mathcal{E} . In this paper we endow all our Clifford modules with the natural grading.

2.3. Dirac operators. The *Dirac operator* $D : \Gamma(M, \mathcal{E}) \rightarrow \Gamma(M, \mathcal{E})$ associated to a Clifford connection $\nabla^\mathcal{E}$ is defined by the following composition

$$\Gamma(M, \mathcal{E}) \xrightarrow{\nabla^\mathcal{E}} \Gamma(M, T^*M \otimes \mathcal{E}) \xrightarrow{c} \Gamma(M, \mathcal{E}). \quad (2.4)$$

In local coordinates, this operator may be written as $D = \sum c(dx^i) \nabla_{\partial_i}^\mathcal{E}$. Note that D sends even sections to odd sections and vice versa: $D : \Gamma(M, \mathcal{E}^\pm) \rightarrow \Gamma(M, \mathcal{E}^\mp)$.

Suppose now that the Clifford module \mathcal{E} is endowed with a Hermitian structure and consider the L^2 -scalar product on the space of sections $\Gamma(M, \mathcal{E})$ defined by the Riemannian metric on M and the Hermitian structure on \mathcal{E} . By [5, Proposition 3.44], the *Dirac operator associated to a Clifford connection $\nabla^\mathcal{E}$ is formally self-adjoint with respect to this scalar product if and only if \mathcal{E} is a self-adjoint Clifford module and $\nabla^\mathcal{E}$ is a Hermitian connection*.

Let \mathcal{L} be a Hermitian line bundle endowed with a Hermitian connection $\nabla^\mathcal{L}$. For each integer $k \geq 0$, we consider the bundle $\mathcal{E} \otimes \mathcal{L}^k$ as a Clifford module with Clifford action $c(a) \otimes 1$. The tensor product connection $\nabla^{\mathcal{E} \otimes \mathcal{L}^k}$ on $\mathcal{E} \otimes \mathcal{L}^k$ is a self-adjoint Clifford connection (cf. Subsection 2.1) and, hence, define a formally self-adjoint Dirac operator $D_k : \Gamma(M, \mathcal{E} \otimes \mathcal{L}^k) \rightarrow \Gamma(M, \mathcal{E} \otimes \mathcal{L}^k)$.

We denote by D_k^\pm the restriction of D_k to the space $\Gamma(M, \mathcal{E}^\pm \otimes \mathcal{L}^k)$.

2.4. A discrete group action. Assume now that a discrete group Γ acts freely on M by orientation preserving isometries and that the quotient manifold $X = M/\Gamma$ is compact. Then Γ acts naturally on $C(M)$ preserving the fiberwise algebra structure.

Suppose that there are given actions of Γ on the bundles \mathcal{E}, \mathcal{L} which cover the action of Γ on M and preserve the Hermitian structures and the connections on \mathcal{E}, \mathcal{L} . We also assume that the Clifford action of $C(M)$ on M is Γ -invariant. Then the Dirac operator D_k commutes with the Γ -action.

It follows from [3, §3], that D_k extends to a self-adjoint unbounded operator on the space $L^2(M, \mathcal{E} \otimes \mathcal{L}^k)$ of square-integrable sections of $\mathcal{E} \otimes \mathcal{L}^k$.

2.5. The vanishing theorem. The curvature $F^\mathcal{L}$ of $\nabla^\mathcal{L}$ is an imaginary valued 2-form on M . If it is non-degenerate at all points of M , then $iF^\mathcal{L}$ is a symplectic form on M , and, hence, defines an orientation of M .

Our first result is the following

Theorem 2.6. *Assume that the curvature $F^\mathcal{L} = (\nabla^\mathcal{L})^2$ of the connection $\nabla^\mathcal{L}$ is non-degenerate at all points of M . If the orientation defined by the symplectic form $iF^\mathcal{L}$ coincides with the original orientation of M , then*

$$\text{Ker } D_k^- = 0 \quad \text{for} \quad k \gg 0. \quad (2.5)$$

Otherwise, $\text{Ker } D_k^+ = 0$ for $k \gg 0$.

The proof is given in Subsection 6.5. For the case when Γ is a trivial group (and, hence, M is a compact manifold) this theorem was established in [8].

Remark 2.7. Since the operators D_k^\pm are Γ -invariant, they are lifts of certain operator $D_{X,k}^\pm$ on X . By the L^2 -index theorem of Atiyah, [3], the Γ -index of D_k^\pm is equal to the usual index of $D_{X,k}^\pm$ (we refer the reader to [3] for the definitions of Γ -dimensions and Γ -index). Combining with our vanishing theorem we obtain

$$\dim_{\Gamma} \text{Ker } D_k^\pm = \dim \text{Ker } D_{X,k}^\pm \quad \text{for any } k \gg 0.$$

Here \dim_{Γ} denotes the Γ -dimension, cf. [3].

Remark 2.8. Theorem 2.6 remains valid if \mathcal{L} is a vector bundle of dimension higher than 1 (cf. [14] for analogous generalization of the Kodaira vanishing theorem). In this case \mathcal{L}^k should be understood as the k -th symmetric power of \mathcal{L} . Also the curvature $F^{\mathcal{L}}$ becomes a 2-form with values in the bundle $\text{End}(\mathcal{L})$ of endomorphisms of \mathcal{L} . We say that it is non-degenerate if, for any $0 \neq \xi \in \mathcal{L}$, the imaginary valued form $\langle F^{\mathcal{L}}\xi, \xi \rangle$ is non-degenerate. In this case, the orientation of M defined by the form $i\langle F^{\mathcal{L}}\xi, \xi \rangle$ is independent of $\xi \neq 0$. If this orientation coincides with (resp. is opposite to) the original orientation of M , then $\text{Ker } D_k^- = 0$ (resp. $\text{Ker } D_k^+ = 0$).

The proof is a combination of the methods of this paper with those of [14]. The details will appear elsewhere.

3. COMPLEX MANIFOLDS. THE L^2 ANALOGUE OF THE ANDREOTTI-GRAUERT THEOREM

In this section we present an L^2 analogue of the Andreotti-Grauert vanishing theorem. This result is a refinement of Theorem 2.6 for the case when M is a complex manifold.

3.1. The reduced L^2 Dolbeault cohomology. Suppose M is a complex manifold endowed with a holomorphic free action of a discrete group Γ , such that the quotient $X = M/\Gamma$ is compact. Let \mathcal{W} be a holomorphic Γ -equivariant vector bundle over M and let \mathcal{L} be a holomorphic Γ -equivariant line bundle over M .

Fix a Γ -invariant Hermitian metric on $TM \otimes \mathbb{C}$ and Γ -invariant Hermitian metrics on the bundles \mathcal{E}, \mathcal{L} . Let $\mathcal{A}^{0,*}(M, \mathcal{E} \otimes \mathcal{L}^k)$ and $L^2\mathcal{A}^{0,*}(M, \mathcal{E} \otimes \mathcal{L}^k)$ denote respectively the spaces of smooth and square-integrable $(0,*)$ -differential forms on M with values in $\mathcal{E} \otimes \mathcal{L}^k$.

Set

$$\begin{aligned} Z^j &= \text{Ker} \left(\bar{\partial} : L^2\mathcal{A}^{0,j}(M, \mathcal{E} \otimes \mathcal{L}^k) \rightarrow L^2\mathcal{A}^{0,j+1}(M, \mathcal{E} \otimes \mathcal{L}^k) \right); \\ B^j &= \text{Im} \left(\bar{\partial} : L^2\mathcal{A}^{0,j-1}(M, \mathcal{E} \otimes \mathcal{L}^k) \rightarrow L^2\mathcal{A}^{0,j}(M, \mathcal{E} \otimes \mathcal{L}^k) \right) \end{aligned}$$

and let \overline{B}^j denote the closure of B^j in $L^2\mathcal{A}^{0,j}(M, \mathcal{E} \otimes \mathcal{L}^k)$.

The (*reduced*) L^2 Dolbeault cohomology of M with coefficients in the bundle $\mathcal{E} \otimes \mathcal{L}^k$ is the quotient space

$$L^2H^j(M, \mathcal{E} \otimes \mathcal{L}^k) = Z^j / \overline{B}^j.$$

Note, that though the L^2 square product depends on the choices of Hermitian metrics on M, \mathcal{E} and \mathcal{L} , the topology of the Hilbert space $L^2\mathcal{A}^{0,*}(M, \mathcal{E} \otimes \mathcal{L}^k)$ does not. So the cohomology $L^2H^j(M, \mathcal{E} \otimes \mathcal{L}^k)$ is essentially independent of the metrics.

Remark 3.2. The reduced L^2 cohomology is isomorphic to the kernel of the Dolbeault-Dirac operator $\sqrt{2}(\bar{\partial} + \bar{\partial}^*) : L^2\mathcal{A}^{0,*}(M, \mathcal{E} \otimes \mathcal{L}^k) \rightarrow L^2\mathcal{A}^{0,*}(M, \mathcal{E} \otimes \mathcal{L}^k)$. If M is a Kähler manifold, then (cf. [5, Proposition 3.67]) the Dolbeault-Dirac operator has the form (2.4) (see Subsection 4.3). This connects the material of this section with Theorem 2.6. See Remark 3.7 for more details.

3.3. The curvature of the Chern connection. Let $\nabla^{\mathcal{L}}$ be the *Chern connection* on \mathcal{L} , i.e., the unique holomorphic connection which preserves the Hermitian metric. Then $\nabla^{\mathcal{L}}$ is preserved by the action of Γ . The curvature $F^{\mathcal{L}}$ of $\nabla^{\mathcal{L}}$ is a Γ -invariant $(1,1)$ -form which is called the *curvature form of the Hermitian metric* $h^{\mathcal{L}}$.

The orientation condition of Theorem 2.6 may be reformulated as follows. Let (z^1, \dots, z^n) be complex coordinates in the neighborhood of a point $x \in M$. The curvature $F^{\mathcal{L}}$ may be written as

$$iF^{\mathcal{L}} = \frac{i}{2} \sum_{i,j} F_{ij} dz^i \wedge d\bar{z}^j.$$

Denote by q the number of negative eigenvalues of the matrix $\{F_{ij}\}$. Clearly, the number q is independent of the choice of the coordinates. We will refer to this number as the *number of negative eigenvalues of the curvature $F^{\mathcal{L}}$ at the point x* . Then the orientation defined by the symplectic form $iF^{\mathcal{L}}$ coincides with the complex orientation of M if and only if q is even.

3.4. The L^2 Andreotti-Grauert theorem. A small variation of the method used in the proof of Theorem 2.6 allows to get a more precise result which depends not only on the parity of q but on q itself. In this way we obtain the following

Theorem 3.5. *Let M be a complex manifold on which a discrete group Γ acts freely so that M/Γ is compact. Let \mathcal{L} be a Γ -equivariant holomorphic line bundle over M . Assume that \mathcal{L} carries a Γ -invariant Hermitian metric whose curvature form $F^{\mathcal{L}}$ has at least q negative and at least p positive eigenvalues at any point $x \in M$. Then, for any Γ -equivariant holomorphic vector bundle \mathcal{W} over M , the cohomology $L^2H^{0,j}(M, \mathcal{W} \otimes \mathcal{L}^k)$ vanishes for $j \neq q, q+1, \dots, n-p$ and $k \gg 0$.*

The proof is given in Subsection 8.2. If Γ is a trivial group, Theorem 3.5 reduces to the classical Andreotti-Grauert vanishing theorem [1, 12].

Contrary to Theorem 2.6, the curvature $F^{\mathcal{L}}$ in Theorem 3.5 needs not be non-degenerate. If $F^{\mathcal{L}}$ is non-degenerate, then the number q of negative eigenvalues of $F^{\mathcal{L}}$ does not depend on the point $x \in M$. Then we obtain the following

Corollary 3.6. *If, in the conditions of Theorem 3.5, the curvature $F^{\mathcal{L}}$ is non-degenerate and has exactly q negative eigenvalues at any point $x \in M$, then $L^2H^{0,j}(M, \mathcal{W} \otimes \mathcal{L}^k)$ vanishes for any $j \neq q$ and $k \gg 0$.*

The most important is the case when the bundle \mathcal{L} is positive, i.e., when the matrix $\{F_{ij}\}$ is positive. In this case Corollary 3.6 generalizes the classical Kodaira vanishing theorem (cf., for example, [5, Theorem 3.72(2)]) to covering manifolds.

Remark 3.7. a. It is interesting to compare Corollary 3.6 with Theorem 2.6 for the case when M is a Kähler manifold. In this case the Dirac operator D_k is equal to the Dolbeault-Dirac operator, cf. [5, Proposition 3.67]. Hence (cf. Remark 3.2), Theorem 2.6 implies that $L^2 H^{0,j}(M, \mathcal{W} \otimes \mathcal{L}^k)$ vanishes when the parity of j is not equal to the parity of q . Corollary 3.6 refines this result.

b. If M is not a Kähler manifold, then the Dirac operator D_k defined by (2.4) is not equal to the Dolbeault-Dirac operator, and the kernel of D_k is not isomorphic to the cohomology $L^2 H^{0,*}(M, \mathcal{W} \otimes \mathcal{L}^k)$. However, we show in Section 8 that the operators D_k and $\sqrt{2}(\bar{\partial} + \bar{\partial}^*)$ have the same asymptotic as $k \rightarrow \infty$. Then the vanishing of the kernel of D_k implies the vanishing of the L^2 -cohomology.

Remark 3.8. As in Remark 2.8, one can generalize Theorem 3.5 to the case when \mathcal{L} is a vector bundle of dimension greater than 1. In this way one obtains, in particular, an L^2 analogue of the Griffiths vanishing theorem [14].

4. ALMOST COMPLEX MANIFOLDS. AN ANALOGUE OF THE ANDREOTTI-GRAUERT THEOREM.

In this section we refine Theorem 2.6 for the case when M is an almost complex manifold and $F^{\mathcal{L}}$ is a $(1,1)$ -form. From another point of view, the results of this section generalize the L^2 Andreotti-Grauert theorem to almost complex manifolds, cf. Remark 4.5.b.

As an application we prove that *semiclassically the spin^c -quantization of an almost complex covering manifold gives an honest Hilbert space* (and not just a virtual one).

4.1. Let M, \mathcal{L}, Γ be as in Subsection 2.4. Assume, in addition, that M is an almost complex $2n$ -dimensional manifold, the action of Γ preserves the almost complex structure J on M , and the curvature $F^{\mathcal{L}}$ is a $(1,1)$ -form on M with respect to J . The later condition implies that, for any $x \in M$ and any basis (e^1, \dots, e^n) of the holomorphic cotangent space $(T^{1,0}M)^*$, one has

$$iF^{\mathcal{L}} = \frac{i}{2} \sum_{i,j} F_{ij} e^i \wedge \bar{e}^j.$$

We denote by q the number of negative eigenvalues of the matrix $\{F_{ij}\}$. As in Subsection 3.3, the orientation of M defined by the symplectic form $iF^{\mathcal{L}}$ depends only on the parity of q . It coincides with the orientation defined by J if and only if q is even.

We fix a Riemannian metric g^{TM} on M , such that the almost complex structure $J : TM \rightarrow TM$ is skew-adjoint with respect to g^{TM} .

4.2. A Clifford action on $\Lambda^j(T^{0,1}M)^*$. Let $\Lambda^j = \Lambda^j(T^{0,1}M)^*$ denote the bundle of $(0,j)$ -forms on M and set

$$\Lambda^+ = \bigoplus_{j \text{ even}} \Lambda^j, \quad \Lambda^- = \bigoplus_{j \text{ odd}} \Lambda^j.$$

Let $\lambda^{1/2}$ be the square root of the complex line bundle $\lambda = \det T^{1,0}M$ and let \mathcal{S} be the spinor bundle over M associated to the Riemannian metric g^{TM} . Although $\lambda^{1/2}$ and \mathcal{S} are defined only locally, unless M is a spin manifold, it is well known (cf. [19, Appendix D]) that the products $\mathcal{S}^\pm \otimes \lambda^{1/2}$ are globally defined and $\Lambda^\pm = \mathcal{S}^\pm \otimes \lambda^{1/2}$. Since the spinor bundle is, by definition,

a Clifford module, the last equality defines a Clifford action of $C(M)$ on Λ , cf. Subsection 2.1. More explicitly, this action may be described as follows: if $f \in \Gamma(M, T^*M)$ decomposes as $f = f^{1,0} + f^{0,1}$ with $f^{1,0} \in \Gamma(M, (T^{1,0}M)^*)$ and $f^{0,1} \in \Gamma(M, (T^{0,1}M)^*)$, then the Clifford action of f on $\alpha \in \Gamma(M, \Lambda)$ equals

$$c(f)\alpha = \sqrt{2} (f^{0,1} \wedge \alpha - \iota(f^{1,0})\alpha). \quad (4.1)$$

Here $\iota(f^{1,0})$ denotes the interior multiplication by the vector field $(f^{1,0})^* \in T^{0,1}M$ dual to the 1-form $f^{1,0}$. This action is self-adjoint with respect to the Hermitian structure on Λ defined by the Riemannian metric g^{TM} on M .

The Levi-Civita connection ∇^{TM} of g^{TM} induces Hermitian connections on $\lambda^{1/2}$ and on \mathcal{S} . Let $\nabla^M = \nabla^S \otimes 1 + 1 \otimes \nabla^{\lambda^{1/2}}$ be the product connection (cf. Subsection 2.1). Then ∇^M is a well-defined Hermitian Clifford connection on bundle Λ .

Note also that the grading $\Lambda = \Lambda^+ \oplus \Lambda^-$ is natural.

4.3. An analogue of the Andreotti-Grauert theorem. Let \mathcal{W} be a Γ -equivariant vector bundle over M endowed with a Γ -invariant Hermitian metric and a Γ -invariant Hermitian connection. Set $\mathcal{E}^\pm = \Lambda^\pm \otimes \mathcal{W}$. Then $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$ is a self-adjoint Clifford module. Let $\nabla^{\mathcal{E}}$ be the product of the connection ∇^M on Λ and a Hermitian connection on \mathcal{W} . Then $\nabla^{\mathcal{E}}$ is a Hermitian Clifford connection on \mathcal{E} . The space $L^2(M, \mathcal{E} \otimes \mathcal{L}^k)$ of square-integrable sections of $\mathcal{E} \otimes \mathcal{L}^k$ coincides with the space $L^2 \mathcal{A}^{0,*}(M, \mathcal{W} \otimes \mathcal{L}^k)$ of square-integrable differential forms of type $(0,*)$ with values in $\mathcal{W} \otimes \mathcal{L}^k$. Let

$$D_k : L^2 \mathcal{A}^{0,*}(M, \mathcal{W} \otimes \mathcal{L}^k) \rightarrow L^2 \mathcal{A}^{0,*}(M, \mathcal{W} \otimes \mathcal{L}^k)$$

denote the Dirac operator corresponding to the tensor product connection on $\mathcal{E} \otimes \mathcal{L}^k$.

For a form $\alpha \in L^2 \mathcal{A}^{0,*}(M, \mathcal{W} \otimes \mathcal{L}^k)$, we denote by $\|\alpha\|$ its L^2 -norm and by α_i its component in $L^2 \mathcal{A}^{0,i}(M, \mathcal{W} \otimes \mathcal{L}^k)$.

Theorem 4.4. *In the situation described above, assume that the matrix $\{F_{ij}\}$ has at least q negative and at least p positive eigenvalues at any point $x \in M$. Then there exists a sequence $\varepsilon_1, \varepsilon_2, \dots$ convergent to zero, such that for any $k \gg 0$ and any $\alpha \in \text{Ker } D_k^2$ one has*

$$\|\alpha_j\| \leq \varepsilon_k \|\alpha\|, \quad \text{for } j \neq q, q+1, \dots, n-p.$$

In particular, if the form $F^{\mathcal{L}}$ is non-degenerate and q is the number of negative eigenvalues of $\{F_{ij}\}$ (which is independent of $x \in M$), then there exists a sequence $\tilde{\varepsilon}_1, \tilde{\varepsilon}_2, \dots$, convergent to zero, such that $\alpha \in \text{Ker } D_k$ implies

$$\|\alpha - \alpha_q\| \leq \tilde{\varepsilon}_k \|\alpha_q\|.$$

Theorem 4.4 is proven in Subsection 7.5. For the case when Γ is a trivial group it was proven in [8].

Remark 4.5. a. Theorem 4.4 implies that, if $F^{\mathcal{L}}$ is non-degenerate, then $\text{Ker } D_k$ is dominated by the component of degree q . If $\alpha \in L^2(M, \mathcal{E}^-)$ (resp. $\alpha \in L^2(M, \mathcal{E}^+)$) and q is even (resp. odd) then $\alpha_q = 0$. So, we obtain the vanishing result of Theorem 2.6 for the case when M is almost complex and $F^{\mathcal{L}}$ is a $(1,1)$ -form.

b. Theorem 4.4 is an analogue of Theorem 3.5. Of course, the cohomology $L^2 H^{0,j}(M, \mathcal{W} \otimes \mathcal{L}^k)$ is not defined if J is not integrable. Moreover, the square D_k^2 of the Dirac operator does not preserve the \mathbb{Z} -grading on $L^2 \mathcal{A}^{0,*}(M, \mathcal{W} \otimes \mathcal{L}^k)$. Hence, one can not hope that the kernel of D_k belongs to $\bigoplus_{j=q}^{n-p} L^2 \mathcal{A}^{0,j}(M, \mathcal{W} \otimes \mathcal{L}^k)$. However, Theorem 4.4 shows, that for any $k \gg 0$ and any $\alpha \in \text{Ker } D_k$, “most of the norm” of α is concentrated in $\bigoplus_{j=q}^{n-p} L^2 \mathcal{A}^{0,j}(M, \mathcal{W} \otimes \mathcal{L}^k)$.

4.6. Positive line bundle. Quantization. Probably the most interesting application of Theorem 4.4 may be obtained by choosing \mathcal{L} to be a positive line bundle. Then Theorem 4.4 (and, in fact, even Theorem 2.6) states that $\text{Ker } D_k^- = 0$. Assume also that \mathcal{W} is a trivial line bundle. In this case, the *index space* $\text{Ker } D_k^+ \ominus \text{Ker } D_k^-$ of D_k plays the role of the “quantum-mechanical space” (or the space of “quantization”) in the scheme of geometric quantization (cf. [21]). The geometric quantization for $k \gg 0$ is called the *semiclassical limit*.

Unfortunately, the index space of D_k is only a *virtual* (or graded) vector space, in general. However, Theorems 2.6 and 4.4 imply that *semiclassically this is an “honest” vector space*. For the case when Γ is the trivial group (so that M is compact) this result was established by Borthwick and Uribe [7, Theorem 2.3].

Note, that Theorem 4.4 implies also that, for large values of k , the quantum mechanical-space $\text{Ker } D_k^+$ is “almost” a subspace of $\mathcal{A}^{0,0}(M, \mathcal{L}^k)$. More precisely, the restriction of the projection $L^2 \mathcal{A}^{0,*}(M, \mathcal{L}^k) \rightarrow L^2 \mathcal{A}^{0,0}(M, \mathcal{L}^k)$ to $\text{Ker } D_k^+$ tends to the identity operator when $k \rightarrow \infty$. This fact is very important in the study of semiclassical properties of quantization, cf. [7, §4].

4.7. Estimate on the Dirac operator. The main ingredient of the proof of Theorem 4.4 (cf. Subsection 7.5) is the following estimate on D_k , which also has an independent interest:

Proposition 4.8. *If the matrix $\{F_{ij}\}$ has at least q negative and at least p positive eigenvalues at any point $x \in M$, then there exists a constant $C > 0$, such that*

$$\|D_k \alpha\| \geq C k^{1/2} \|\alpha\|,$$

for any $k \gg 0$, $j \neq q, q+1, \dots, n-p$ and $\alpha \in L^2 \mathcal{A}^{0,j}(M, \mathcal{W} \otimes \mathcal{L}^k) \cap \mathcal{A}^{0,j}(M, \mathcal{W} \otimes \mathcal{L}^k)$.

The proof is given in Subsection 7.2.

5. APPLICATION TO THE REPRESENTATION THEORY

In this section we explain very briefly how Theorems 2.6 and 3.5 can be applied to the study of homogeneous vector bundles. In particular we recover a vanishing theorem which was originally conjectured by Langlands [18] and proven by Griffiths and Schmid [15]. We also indicate how one can use certain generalizations of Theorem 2.6 to get a new proof of the vanishing theorem of Atiyah and Schmid [2, Th. 5.20]. The details will appear in a separate paper.

5.1. New proof of a theorem of Griffiths and Schmid. Let G be a connected non-compact real semi-simple Lie group and assume that it has a compact Cartan subgroup H . Let $K \supset H$ be a maximal compact subgroup of G . Let $\mathfrak{g}, \mathfrak{k}, \mathfrak{h}$ denote the Lie algebras of G, K, H and let $\mathfrak{g}_{\mathbb{C}}, \mathfrak{k}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}$ denote their complexifications.

Denote by Δ the set of roots for $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$. It decomposes as a disjoint union $\Delta = \Delta_c \cup \Delta_n$, where Δ_c and Δ_n are respectively the set of compact and non-compact roots with respect to $K \subset G$. Choose a system $P \subset \Delta$ of positive roots.

Let $M = G/H$ and let $\mathcal{L}_\lambda \rightarrow M$ be the line bundle over M induced by the character λ of H . It is well known (cf. [15, §1]) that the choice of positive root system P defines a complex structure on M and that \mathcal{L}_λ has a natural structure of a holomorphic line bundle over M .

By a theorem of Borel [6], there exists a discrete subgroup $\Gamma \subset G$ which acts freely on M and such that the quotient space $\Gamma \backslash M = \Gamma \backslash G/H$ is compact. This allows us to apply Theorem 3.5 to the study of L^2 cohomology of \mathcal{L}_λ . As a result we obtain a new proof of the following theorem, which was originally conjectured by Langlands [18] and proven by Griffiths and Schmid [15, Th. 7.8].

Theorem 5.2. *Let (\cdot, \cdot) denote the scalar product on \mathfrak{h}^* induced by the Cartan-Killing form on \mathfrak{h} . Suppose λ is a character of H , such that $(\lambda, \alpha) \neq 0$ for any $\alpha \in \Delta$ and set*

$$\iota(\lambda) = \#\{\alpha \in P \cap \Delta_c : (\lambda, \alpha) < 0\} + \#\{\alpha \in P \cap \Delta_n : (\lambda, \alpha) > 0\}.$$

Let ρ denotes the half-sum of the positive roots of $(\mathfrak{g}, \mathfrak{h})$. There exists an integer m such that, if $k \geq m$, then

$$L^2 H^{0,j}(M, \mathcal{L}_{k\lambda}) = 0 \quad \text{for } j \neq \iota(\lambda + \rho).$$

Proof. By [15, Th. 4.17D], the bundle \mathcal{L} possesses a Chern connection whose curvature is non-degenerate and has exactly $\iota(\lambda + \rho)$ negative eigenvalue. Theorem 5.2 follows now from Corollary 3.6. \square

Remark 5.3. 1. The significance of Theorem 5.2 is that it allows to prove (cf. [15]) that, for $j = \iota(\lambda + \rho), k \gg 0$, the space $L^2 H^{0,j}(M, \mathcal{L}_{k\lambda})$ is an irreducible discrete series representation of G . In this way “most” of the discrete series representation may be obtained.

2. Theorem 5.2 may be considerably improved. In particular (cf. [15, Th. 7.8]) there exists a constant b , depending only on G and H , such that

$$L^2 H^{0,j}(M, \mathcal{L}_\lambda) = 0 \quad \text{for } j \neq \iota(\lambda + \rho).$$

whenever λ satisfies $|(\lambda, \alpha)| > b$ for every $\alpha \in \Delta$.

3. Let $G_{\mathbb{C}}$ be a complex form of G and suppose that $B \subset H$ is a Borel subgroup of $G_{\mathbb{C}}$, such that $V = G \cap B$ is compact. Let L_λ denote the irreducible V module with highest weight λ and let $\mathcal{L}_\lambda = G \times_V L_\lambda$ be the corresponding homogeneous vector bundle over $M = G/V$. Using the generalization of Theorem 3.5 discussed in Remark 3.8, one can show that Theorem 5.2 remains true in this case. In this form the theorem is proven in [15].

5.4. A vanishing theorem of Atiyah and Schmid. Let L_λ denote the irreducible K module with highest weight λ and let $\mathcal{L}_\lambda = G \times_K L_\lambda$ be the corresponding homogeneous vector bundle over $M = G/K$. For simplicity, assume also that M possesses a G -equivariant spinor bundle $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$. Then (cf. [2]) there is a canonically defined Dirac operator $D_\lambda^\pm : L^2(M, \mathcal{L}_\lambda \otimes \mathcal{S}^\pm) \rightarrow L^2(M, \mathcal{L}_\lambda \otimes \mathcal{S}^\mp)$. Using the generalization of Theorem 2.6 discussed in Remark 2.8, one can show

that $\text{Ker } D_\lambda = 0$ if λ is *sufficiently non-singular* (i.e. if $|(\lambda, \alpha)| \gg 0$ for any $\alpha \in \Delta$). This is a particular case of [2, Th. 5.20].

6. PROOF OF THE VANISHING THEOREM FOR THE HALF-KERNEL OF A DIRAC OPERATOR

In this section we present a proof of Theorem 2.6 based on Propositions 6.3 and 6.4, which will be proved in the following sections.

The idea of the proof is to study the large k behavior of the square D_k^2 of the Dirac operator.

6.1. The operator \tilde{J} . We need some additional definitions. Recall that $F^\mathcal{L}$ denotes the curvature of the connection $\nabla^\mathcal{L}$. In this subsection we do not assume that $F^\mathcal{L}$ is non-degenerate. For $x \in M$, define the skew-symmetric linear map $\tilde{J}_x : T_x M \rightarrow T_x M$ by the formula

$$iF^\mathcal{L}(v, w) = g^{TM}(v, \tilde{J}_x w), \quad v, w \in T_x M.$$

The eigenvalues of \tilde{J}_x are purely imaginary. Define

$$\tau(x) = \text{Tr}^+ \tilde{J}_x := \mu_1 + \cdots + \mu_l, \quad m(x) = \min_j \mu_j(x). \quad (6.1)$$

where $i\mu_j$, $j = 1, \dots, l$ are the eigenvalues of \tilde{J}_x for which $\mu_j > 0$. Note that $m(x)$ is well defined and positive if the curvature $F^\mathcal{L}$ does not vanishes at the point $x \in M$.

6.2. Estimate on D_k^2 . Consider the *rough (or metric) Laplacian*

$$\Delta_k := (\nabla^{\mathcal{E} \otimes \mathcal{L}^k})^* \nabla^{\mathcal{E} \otimes \mathcal{L}^k},$$

where $(\nabla^{\mathcal{E} \otimes \mathcal{L}^k})^*$ denote the formal adjoint of the covariant derivative

$$\nabla^{\mathcal{E} \otimes \mathcal{L}^k} : C_0^\infty(M, \mathcal{E} \otimes \mathcal{L}^k) \rightarrow C_0^\infty(M, \mathcal{E} \otimes \mathcal{L}^k \otimes T^* M).$$

Here, as usual, C_0^∞ denotes the space of sections with compact support.

Since Δ_k is an elliptic Γ -invariant operator, it follows from [3, Proposition 3.1] that it is a self-adjoint operator.

Our estimate on the square D_k^2 of the Dirac operator is obtained in two steps: first we compare it to the rough Laplacian Δ_k and then we estimate the large k behavior of Δ_k . These two steps are the subject of the following two propositions.

Proposition 6.3. a. *For any integer k , the difference $D_k^2 - \Delta_k$ is a bounded operator on $L^2(M, \mathcal{E} \otimes \mathcal{L}^k)$.*

b. Supposed that the differential form $F^\mathcal{L}$ is non-degenerate. If the orientation defined on M by the symplectic form $iF^\mathcal{L}$ coincides with (resp. is opposite to) the given orientation of M , then there exists a constant C such that, for any $s \in L^2(M, \mathcal{E}^- \otimes \mathcal{L}^k)$ (resp. for any $s \in L^2(M, \mathcal{E}^+ \otimes \mathcal{L}^k)$), one has an estimate

$$\langle (D_k^2 - \Delta_k) s, s \rangle \geq -k \langle (\tau(x) - 2m(x)) s, s \rangle - C \|s\|^2.$$

Here $\langle \cdot, \cdot \rangle$ denotes the L^2 -scalar product on the space of sections and $\|\cdot\|$ is the norm corresponding to this scalar product.

The proposition is proven in Subsection 9.3 using the Lichnerowicz formula (9.3).

Set

$$\text{Dom}(D_k^2) := \left\{ s \in L^2(M, \mathcal{E} \otimes \mathcal{L}^k) : D_k^2 s \in L^2(M, \mathcal{E} \otimes \mathcal{L}^k) \right\}.$$

where $D_k^2 s$ is understood in the sense of distributions. In the next proposition we do not assume that $F^\mathcal{L}$ is non-degenerate.

Proposition 6.4. *Suppose that $F^\mathcal{L}$ does not vanish at any point $x \in M$. For any $\varepsilon > 0$, there exists a constant C_ε such that, for any $k \in \mathbb{Z}$ and any $s \in \text{Dom}(D_k^2)$ one has*

$$\langle \Delta_k s, s \rangle \geq k \langle (\tau(x) - \varepsilon) s, s \rangle - C_\varepsilon \|s\|^2. \quad (6.2)$$

Proposition 6.4 is proven in Section 10. For the case when Γ is a trivial group (so that M is compact) it was essentially proven in [7, Theorem 2.1] (see also [8, Proposition 4.4]).

6.5. Proof of Theorem 2.6. Assume that the orientation defined by $iF^\mathcal{L}$ coincides with the given orientation of M and $s \in L^2(M, \mathcal{E}^- \otimes \mathcal{L})$, or that the orientation defined by $iF^\mathcal{L}$ is opposite to the given orientation of M and $s \in L^2(M, \mathcal{E}^+ \otimes \mathcal{L})$. By Proposition 6.3,

$$\langle D_k^2 s, s \rangle \geq \langle \Delta_k s, s \rangle - k \langle (\tau(x) - 2m(x)) s, s \rangle - C \|s\|^2. \quad (6.3)$$

Choose

$$0 < \varepsilon < 2 \min_{x \in M} m(x)$$

and set

$$C' = 2 \min_{x \in M} m(x) - \varepsilon > 0.$$

Assume now that $D_k^2 s \in L^2(M, \mathcal{E} \otimes \mathcal{L}^k)$. Then, it follows from Proposition 6.3.a, that $\Delta_k s \in L^2(M, \mathcal{E} \otimes \mathcal{L}^k)$. Using (6.2) and (6.3), we obtain

$$\langle D_k^2 s, s \rangle \geq (kC' - (C + C_\varepsilon)) \|s\|^2. \quad (6.4)$$

Thus, for $k > (C + C_\varepsilon)/C'$, we have $\langle D_k^2 s, s \rangle > 0$. Hence, $D_k s \neq 0$. \square

7. PROOF OF THE ANDREOTTI-GRAUERT-TYPE THEOREM FOR ALMOST COMPLEX MANIFOLDS

In this section we prove Theorem 4.4 and Proposition 4.8. The proof is very similar to the proof of Theorem 2.6 (cf. Section 6). It is based on Proposition 6.4 and the following refinement of Proposition 6.3:

Proposition 7.1. *Assume that the matrix $\{F_{ij}\}$ (cf. Subsection 3.3) has at least q negative eigenvalues at any point $x \in M$. For any $x \in M$, we denote by $m_q(x) > 0$ the minimal positive number, such that at least q of the eigenvalues of $\{F_{ij}\}$ do not exceed $-m_q$. Then there exists a constant C such that*

$$\langle (D_k^2 - \Delta_k) \alpha, \alpha \rangle \geq -k \langle (\tau(x) - 2m_q(x)) \alpha, \alpha \rangle - C \|\alpha\|^2$$

for any $j = 0, \dots, q-1$ and any $\alpha \in L^2 \mathcal{A}^{0,j}(M, \mathcal{W} \otimes \mathcal{L}^k)$.

The proposition is proven in Subsection 9.12.

7.2. Proof of Proposition 4.8. Choose $0 < \varepsilon < 2 \min_{x \in M} m_q(x)$ and set

$$C' = 2 \min_{x \in M} m_q(x) - \varepsilon.$$

Fix $j = 0, \dots, q-1$ and $\alpha \in L^2 \mathcal{A}^{0,j}(M, \mathcal{W} \otimes \mathcal{L}^k)$ such that $D_k^2 \alpha \in L^2 \mathcal{A}^{0,*}(M, \mathcal{W} \otimes \mathcal{L}^k)$. By Proposition 6.3.a, $\Delta_k \alpha \in L^2 \mathcal{A}^{0,*}(M, \mathcal{W} \otimes \mathcal{L}^k)$. It follows from Propositions 6.4 and 7.1, that

$$\langle D_k^2 \alpha, \alpha \rangle \geq k C' \|\alpha\|^2 - (C + C_\varepsilon) \|\alpha\|^2.$$

Hence, for any $k > 2(C + C_\varepsilon)/C'$, we have

$$\|D_k \alpha\|^2 = \langle D_k^2 \alpha, \alpha \rangle \geq \frac{k C'}{2} \|\alpha\|^2.$$

This proves Proposition 4.8 for $j = 0, \dots, q-1$. The statement for $j = n-p+1, \dots, n$ may be proven by a verbatim repetition of the above arguments, using a natural analogue of Proposition 7.1. (Alternatively, the statement for $j = n-p+1, \dots, n$ may be obtained as a formal consequence of the statement for $j = 0, \dots, q-1$ by considering M with an opposite almost complex structure). \square

7.3. If the manifold M is not Kähler, then the operator D_k^2 does not preserve the \mathbb{Z} -grading on $\mathcal{A}^{0,*}(M, \mathcal{W} \otimes \mathcal{L}^k)$. However, the next proposition shows that the *mixed degree operator* $\alpha_i \mapsto (D_k^2 \alpha_i)_j$ is, in a certain sense, small.

Proposition 7.4. Set $\text{Dom}^i(D_k^2) = \{\alpha \in L^2 \mathcal{A}^{0,i}(M, \mathcal{W} \otimes \mathcal{L}^k) : D_k^2 \alpha \in L^2 \mathcal{A}^{0,*}(M, \mathcal{W} \otimes \mathcal{L}^k)\}$, where $D_k^2 \alpha$ is understood in the sense of distributions. There exists a sequence $\delta_1, \delta_2, \dots$, such that $\lim_{k \rightarrow \infty} \delta_k = 0$ and

$$|\langle D_k^2 \alpha, \beta \rangle| \leq \delta_k \langle D_k^2 \alpha, \alpha \rangle + \delta_k \langle D_k^2 \beta, \beta \rangle + \delta_k k \|\alpha\|^2 + \delta_k k \|\beta\|^2,$$

for any $i \neq j$ and any $\alpha \in \text{Dom}^i(D_k^2), \beta \in \text{Dom}^j(D_k^2)$.

The proof of the proposition, based on the Lichnerowicz formula and [11], is given in Subsection 9.13.

7.5. Proof of Theorem 4.4. Let $\alpha \in \text{Ker } D_k$ and fix $j \notin q, q+1, \dots, n-p$. Set $\beta = \alpha - \alpha_j$. Then

$$0 = \|D_k \alpha\|^2 = \|D_k \alpha_j\|^2 + 2 \operatorname{Re} \langle D_k \alpha_j, D_k \beta \rangle + \|D_k \beta\|^2.$$

Hence, it follows from Proposition 7.4, that

$$0 \geq (1 - 2\delta_k) \|D_k \alpha_j\|^2 + (1 - 2\delta_k) \|D_k \beta\|^2 - 2\delta_k k \|\alpha_j\|^2 - 2\delta_k k \|\beta\|^2. \quad (7.1)$$

If we assume now that k is large enough, so that $1 - 2\delta_k > 0$, then we obtain from (7.1) and Proposition 4.8 that

$$((1 - 2\delta_k) C^2 k - 2\delta_k k) \|\alpha_j\|^2 \leq 2\delta_k k \|\beta\|^2 \leq 2\delta_k k \|\alpha\|^2.$$

Thus

$$\|\alpha_j\|^2 \leq \frac{2\delta_k}{(1 - 2\delta_k) C^2 - 2\delta_k} \|\alpha\|^2.$$

Hence, Theorem 4.4 holds with $\varepsilon_k = \sqrt{\frac{2\delta_k}{(1-2\delta_k)C^2-2\delta_k}}$. \square

8. PROOF OF THE L^2 ANDREOTTI-GRAUERT THEOREM

In this section we use the results of Section 4 in order to prove our L^2 version of the Andreotti-Grauert (Theorem 3.5).

Note, first, that, if the manifold M is Kähler, then the L^2 Andreotti-Grauert theorem follows directly from Theorem 4.4. Indeed, in this case, the Dirac operator D_k is equal to the Dolbeault-Dirac operator $\sqrt{2}(\bar{\partial} + \bar{\partial}^*)$, cf. [5, Proposition 3.67]. Hence, the restriction of the kernel of D_k to $L^2\mathcal{A}^{0,j}(M, \mathcal{O}(\mathcal{W} \otimes \mathcal{L}^k))$ is isomorphic to the reduced L^2 -cohomology $L^2H^j(M, \mathcal{O}(\mathcal{W} \otimes \mathcal{L}^k))$.

In general, $D_k \neq \sqrt{2}(\bar{\partial} + \bar{\partial}^*)$. However, the following proposition shows that those two operators have the same “large k behavior”.

Proposition 8.1. *Set $\mathcal{E} = \Lambda(T^{0,1}M) \otimes \mathcal{W}$. In the conditions of Theorem 3.5, there exists a Γ -invariant bundle map $A \in \text{End}(\mathcal{E}) \subset \text{End}(\mathcal{E} \otimes \mathcal{L}^k)$, independent of k , such that*

$$\sqrt{2}(\bar{\partial} + \bar{\partial}^*) = D_k + A. \quad (8.1)$$

Proof. Choose a holomorphic section $e(x)$ of \mathcal{L} over an open set $U \subset M$. It defines a section $e^k(x)$ of \mathcal{L}^k over U and, hence, a holomorphic trivialization

$$U \times \mathbb{C} \xrightarrow{\sim} \mathcal{L}^k, \quad (x, \phi) \mapsto \phi \cdot e^k(x) \in \mathcal{L}^k \quad (8.2)$$

of the bundle \mathcal{L}^k over U . Similarly, the bundles \mathcal{W} and $\mathcal{W} \otimes \mathcal{L}^k$ may be identified over U by the formula

$$w \mapsto w \otimes e^k. \quad (8.3)$$

Let $h^{\mathcal{L}}$ and $h^{\mathcal{W}}$ denote the Hermitian fiberwise metrics on the bundles \mathcal{L} and \mathcal{W} respectively. Let $h^{\mathcal{W} \otimes \mathcal{L}^k}$ denote the Hermitian metric on $\mathcal{W} \otimes \mathcal{L}^k$ induced by the metrics $h^{\mathcal{L}}, h^{\mathcal{W}}$. Set

$$f(x) := |e(x)|^2, \quad x \in U,$$

where $|\cdot|$ denotes the norm defined by the metric $h^{\mathcal{L}}$. Under the isomorphism (8.3) the metric $h^{\mathcal{W} \otimes \mathcal{L}^k}$ corresponds to the metric

$$h_k(\cdot, \cdot) = f^k h^{\mathcal{W}}(\cdot, \cdot) \quad (8.4)$$

on \mathcal{W} .

By [5, p. 137], the connection $\nabla^{\mathcal{L}}$ on \mathcal{L} corresponds under the trivialization (8.2) to the operator

$$\Gamma(U, \mathbb{C}) \rightarrow \Gamma(U, T^*U \otimes \mathbb{C}); \quad s \mapsto ds + kf^{-1}\partial f \wedge s.$$

Similarly, the connection on $\mathcal{E} \otimes \mathcal{L}^k = \Lambda(T^{0,1}M)^* \otimes \mathcal{W} \otimes \mathcal{L}^k$ corresponds under the isomorphism (8.3) to the connection

$$\nabla_k : \alpha \mapsto \nabla^{\mathcal{E}}\alpha + kf^{-1}\partial f \wedge \alpha, \quad \alpha \in \Gamma(U, \Lambda(T^{0,1}U)^* \otimes \mathcal{W}|_U)$$

on $\mathcal{E}|_U$. It follows now from (4.1) and (2.4) that the Dirac operator D_k corresponds under (8.3) to the operator

$$\tilde{D}_k : \alpha \mapsto D_0\alpha - \sqrt{2}kf^{-1}\iota(\partial f)\alpha, \quad \alpha \in \mathcal{A}^{0,*}(U, \mathcal{W}|_U). \quad (8.5)$$

Here $\iota(\partial f)$ denotes the contraction with the vector field $(\partial f)^* \in T^{0,1}M$ dual to the 1-form ∂f , and D_0 stands for the Dirac operator on the bundle $\mathcal{E} = \mathcal{E} \otimes \mathcal{L}^0$.

Let $\bar{\partial}_k^* : \mathcal{A}^{0,*}(U, \mathcal{W}|_U) \rightarrow \mathcal{A}^{0,*-1}(U, \mathcal{W}|_U)$ denote the formal adjoint of the operator $\bar{\partial}$ with respect to the scalar product on $\mathcal{A}^{0,*}(U, \mathcal{W}|_U)$ determined by the Hermitian metric h_k on \mathcal{W} and the Riemannian metric on M . Then, it follows from (8.4), that

$$\bar{\partial}_k^* = \bar{\partial} + kf^{-1}\iota(\partial f). \quad (8.6)$$

By (8.5) and (8.6), we obtain

$$\sqrt{2}(\bar{\partial} + \bar{\partial}_k^*) - \tilde{D}_k = \sqrt{2}(\bar{\partial} + \bar{\partial}_0^*) - D_0.$$

Set $A = \sqrt{2}(\bar{\partial} + \bar{\partial}_0^*) - D_0$. By [13, Lemma 5.5], A is a zero order operator, i.e. $A \in \text{End}(\mathcal{E})$ (note that our definition of the Clifford action on $\Lambda(T^{0,1}M)^*$ and, hence, of the Dirac operator defers from [13] by a factor of $\sqrt{2}$).

Since both operators D_0 and $\sqrt{2}(\bar{\partial} + \bar{\partial}_0^*) - D_0$ are Γ -invariant, so is A . \square

8.2. Proof of Theorem 3.5. Since the bundle map $A \in \text{End}(\mathcal{E})$ defined in Proposition 8.1 is Γ -invariant, it defines a bounded operator $L^2\mathcal{A}^{0,*}(M, \mathcal{E} \otimes \mathcal{L}^k) \rightarrow L^2\mathcal{A}^{0,*}(M, \mathcal{E} \otimes \mathcal{L}^k)$. Let

$$\|A\| = \sup_{\|\alpha\|=1} \|A\alpha\|, \quad \alpha \in L^2\mathcal{A}^{0,*}(M, \mathcal{E} \otimes \mathcal{L}^k)$$

be the norm of this operator. By Proposition 4.8, there exists a constant $C > 0$ such that

$$\|D_k \alpha\| \geq Ck^{1/2} \|\alpha\|,$$

for any $k \gg 0$, $j \neq q, q+1, \dots, n-p$ and $\alpha \in L^2\mathcal{A}^{0,j}(M, \mathcal{W} \otimes \mathcal{L}^k) \cap \mathcal{A}^{0,j}(M, \mathcal{W} \otimes \mathcal{L}^k)$. Then, if $k > \|A\|^2/C^2$, we have

$$\|\sqrt{2}(\bar{\partial} + \bar{\partial}_0^*)\alpha\| = \|(D_k + A)\alpha\| \geq \|D_k\alpha\| - \|A\|\|\alpha\| \geq \left(Ck^{1/2} - \|A\|\right) \|\alpha\| > 0,$$

for any $j \neq q, q+1, \dots, n-p$ and $0 \neq \alpha \in L^2\mathcal{A}^{0,j}(M, \mathcal{W} \otimes \mathcal{L}^k)$. Hence, the restriction of the kernel of the Dolbeault-Dirac operator to the space $L^2\mathcal{A}^{0,j}(M, \mathcal{E} \otimes \mathcal{L}^k)$ vanishes for $j \neq q, q+1, \dots, n-p$. \square

9. THE LICHNEROWICZ FORMULA. PROOF OF PROPOSITIONS 6.3, 7.1 AND 7.4

In this section we use the Lichnerowicz formula (cf. Subsection 9.3) to prove the Propositions 6.3, 7.1 and 7.4.

9.1. The curvature of a Clifford connection. Before formulating the Lichnerowicz formula, we need some more information about Clifford modules and Clifford connections (cf. [5, Section 3.3]).

Let $\nabla^{\mathcal{E}}$ be a Clifford connection on a Clifford module \mathcal{E} and let $F^{\mathcal{E}} = (\nabla^{\mathcal{E}})^2 \in \mathcal{A}^2(M, \text{End}(\mathcal{E}))$ denote the curvature of $\nabla^{\mathcal{E}}$.

Let $\text{End}_{C(M)}(\mathcal{E})$ denote the bundle of endomorphisms of \mathcal{E} commuting with the action of the Clifford bundle $C(M)$. Then the bundle $\text{End}(\mathcal{E})$ of all endomorphisms of \mathcal{E} is naturally isomorphic to the tensor product

$$\text{End}(\mathcal{E}) \cong C(M) \otimes \text{End}_{C(M)}(\mathcal{E}). \quad (9.1)$$

By Proposition 3.43 of [5], $F^{\mathcal{E}}$ decomposes with respect to (9.1) as

$$F^{\mathcal{E}} = R^{\mathcal{E}} + F^{\mathcal{E}/\mathcal{S}}, \quad R^{\mathcal{E}} \in \mathcal{A}^2(M, C(M)), \quad F^{\mathcal{E}/\mathcal{S}} \in \mathcal{A}^2(M, \text{End}_{C(M)}(\mathcal{E})). \quad (9.2)$$

In this formula, $F^{\mathcal{E}/\mathcal{S}}$ is an invariant of $\nabla^{\mathcal{E}}$ called the *twisting curvature* of \mathcal{E} , and $R^{\mathcal{E}}$ is determined by the Riemannian curvature R of M . If (e_1, \dots, e_{2n}) is an orthonormal frame of the tangent space $T_x M$, $x \in M$ and (e^1, \dots, e^{2n}) is the dual frame of the cotangent space $T^* M$, then

$$R^{\mathcal{E}}(e_i, e_j) = \frac{1}{4} \sum_{k,l} \langle R(e_i, e_j) e_k, e_l \rangle c(e^k) c(e^l).$$

Remark 9.2. Assume that \mathcal{S} is a spinor bundle ([5, §3.3]), $\mathcal{E} = \mathcal{W} \otimes \mathcal{S}$ and $\nabla^{\mathcal{E}}$ is given by the tensor product of a Hermitian connection on \mathcal{W} and the Levi-Civita connection on \mathcal{S} . Then $\mathcal{A}(M, \text{End}_{C(M)}(\mathcal{E})) \cong \mathcal{A}(M, \text{End}(\mathcal{W}))$ and the twisting curvature $F^{\mathcal{E}/\mathcal{S}}$ is equal to the curvature $F^{\mathcal{W}} = (\nabla^{\mathcal{W}})^2$ via this isomorphism (cf. [5, p. 121]). This explains why $F^{\mathcal{E}/\mathcal{S}}$ is called the twisting curvature.

9.3. The Lichnerowicz formula. Let \mathcal{E} be a Clifford module endowed with a Hermitian structure and let $D : \Gamma(M, \mathcal{E}) \rightarrow \Gamma(M, \mathcal{E})$ be a self-adjoint Dirac operator associated to a Hermitian Clifford connection $\nabla^{\mathcal{E}}$. Consider the rough Laplacian (cf. Subsection 6.2)

$$\Delta^{\mathcal{E}} = (\nabla^{\mathcal{E}})^* \nabla^{\mathcal{E}} : \Gamma(M, \mathcal{E}) \rightarrow \Gamma(M, \mathcal{E}),$$

where $(\nabla^{\mathcal{E}})^*$ denotes the formal adjoint of $\nabla^{\mathcal{E}} : \Gamma(M, \mathcal{E}) \rightarrow \Gamma(M, T^* M \otimes \mathcal{E})$. By [3, Proposition 3.1], the operator $\Delta^{\mathcal{E}}$ is self-adjoint.

The following *Lichnerowicz formula* (cf. [5, Theorem 3.52]) plays a crucial role in our proof of vanishing theorems:

$$D^2 = \Delta^{\mathcal{E}} + \mathbf{c}(F^{\mathcal{E}/\mathcal{S}}) + \frac{r_M}{4}. \quad (9.3)$$

Here r_M stands for the scalar curvature of M , $F^{\mathcal{E}/\mathcal{S}}$ is the twisting curvature of $\nabla^{\mathcal{E}}$ and

$$\mathbf{c}(F) := \sum_{i < j} F(e_i, e_j) c(e^i) c(e^j), \quad F \in \mathcal{A}^2(M, \text{End}(\mathcal{E})), \quad (9.4)$$

where (e_1, \dots, e_{2n}) is an orthonormal frame of the tangent space to M , and (e^1, \dots, e^{2n}) is the dual frame of the cotangent space.

Let \mathcal{L} be a Hermitian line bundle over M endowed with a Hermitian connection $\nabla^{\mathcal{L}}$ and let $\nabla_k = \nabla^{\mathcal{E} \otimes \mathcal{L}^k}$ denote the product connection (cf. (2.1)) on $\mathcal{E} \otimes \mathcal{L}^k$. It is a Hermitian Clifford connection on $\mathcal{E} \otimes \mathcal{L}^k$. The twisting curvature of ∇_k is given by

$$F^{(\mathcal{E} \otimes \mathcal{L}^k)/\mathcal{S}} = kF^{\mathcal{L}} + F^{\mathcal{E}/\mathcal{S}}. \quad (9.5)$$

We denote by D_k and Δ_k the Dirac operator and the rough Laplacian associated to this connection. By (9.5), it follows from the Lichnerowicz formula (9.3), that

$$D_k^2 = \Delta_k + k\mathbf{c}(F^{\mathcal{L}}) + A, \quad (9.6)$$

where $F^{\mathcal{L}} = (\nabla^{\mathcal{L}})^2$ is the curvature of $\nabla^{\mathcal{L}}$ and

$$A := \mathbf{c}(F^{\mathcal{E}/\mathcal{S}}) + \frac{r_M}{4} \in \text{End}(\mathcal{E}) \subset \text{End}(\mathcal{E} \otimes \mathcal{L}^k) \quad (9.7)$$

is independent of \mathcal{L} and k .

9.4. Calculation of $\mathbf{c}(F^{\mathcal{L}})$. To compare D_k^2 with the Laplacian Δ_k we now need to calculate the operator $\mathbf{c}(F^{\mathcal{L}}) \in \text{End}(\mathcal{E}) \subset \text{End}(\mathcal{E} \otimes \mathcal{L}^k)$. This may be reformulated as the following problem of linear algebra.

Let V be an oriented Euclidean vector space of real dimension $2n$ and let V^* denote the dual vector space. We denote by $C(V)$ the Clifford algebra of V^* . Let E be a module over $C(V)$. We will assume that E is endowed with a Hermitian scalar product such that the operator $c(v) : E \rightarrow E$ is skew-symmetric for any $v \in V^*$. In this case we say that E is a *self-adjoint* Clifford module over V .

The space E possesses a *natural grading* $E = E^+ \oplus E^-$, where E^+ and E^- are the eigenspaces of the chirality operator with eigenvalues $+1$ and -1 respectively, cf. Subsection 2.2.

In our applications V is the tangent space $T_x M$ to M at a point $x \in M$ and E is the fiber of \mathcal{E} over x .

Let F be an imaginary valued antisymmetric bilinear form on V . Then F may be considered as an element of $V^* \wedge V^*$. We need to estimate the operator $\mathbf{c}(F) \in \text{End}(E)$. Here $\mathbf{c} : \Lambda V^* \rightarrow C(V)$ defined exactly as in (9.4) (cf. [5, §3.1]).

Let us define the skew-symmetric linear map $\tilde{J} : V \rightarrow V$ by the formula

$$iF(v, w) = \langle v, \tilde{J}w \rangle, \quad v, w \in V.$$

The eigenvalues of \tilde{J} are purely imaginary. Let $\mu_1 \geq \dots \geq \mu_l > 0$ be the positive numbers such that $\pm i\mu_1, \dots, \pm i\mu_l$ are all the non-zero eigenvalues of \tilde{J} . Set

$$\tau = \text{Tr}^+ \tilde{J} := \mu_1 + \dots + \mu_l, \quad m = \min_j \mu_j.$$

Clearly, in the conditions of Proposition 6.3, the bundle map A , defined in (9.7), is invariant with respect to Γ . Thus it defines a bounded operator $L^2(M, \mathcal{E} \otimes \mathcal{L}^k) \rightarrow L^2(M, \mathcal{E} \otimes \mathcal{L}^k)$. Hence, by the Lichnerowicz formula (9.3), Proposition 6.3 is equivalent to the following

Proposition 9.5. *Suppose that the bilinear form F is non-degenerate. Then it defines an orientation of V . If this orientation coincides with (resp. is opposite to) the given orientation of V ,*

then the restriction of $\mathbf{c}(F)$ onto E^- (resp. E^+) is greater than $-(\tau - 2m)$, i.e., for any $\alpha \in E^-$ (resp. $\alpha \in E^+$)

$$\langle \mathbf{c}(F)\alpha, \alpha \rangle \geq -(\tau - 2m) \|\alpha\|^2.$$

We will prove the proposition in Subsection 9.11 after introducing some additional constructions. Since we need these constructions also for the proof of Proposition 7.1, we do not assume that F is non-degenerate unless this is stated explicitly.

9.6. A choice of a complex structure on V . By the Darboux theorem (cf. [4, Theorem 1.3.2]), one can choose an orthonormal basis f^1, \dots, f^{2n} of V^* , which defines the positive orientation of V (i.e., $f^1 \wedge \dots \wedge f^{2n}$ is a positive volume form on V) and such that

$$iF_x^{\mathcal{L}} = \sum_{j=1}^l r_j f^j \wedge f^{j+n}, \quad (9.8)$$

for some integer $l \leq n$ and some non-zero real numbers r_j . We can and we will assume that $|r_1| \geq |r_2| \geq \dots \geq |r_l|$.

Let f_1, \dots, f_{2n} denote the dual basis of V .

Remark 9.7. If the vector space V is endowed with a complex structure $J : V \rightarrow V$ compatible with the metric (i.e., $J^* = -J$) and such that F is a $(1,1)$ form with respect to J , then the basis f_1, \dots, f_{2n} can be chosen so that $f_{j+n} = Jf_j$, $i = 1, \dots, n$.

Let us define a complex structure $J : V \rightarrow V$ on V by the condition $f_{i+n} = Jf_i$, $i = 1, \dots, n$. Then, the complexification of V splits into the sum of its holomorphic and anti-holomorphic parts

$$V \otimes \mathbb{C} = V^{1,0} \oplus V^{0,1},$$

on which J acts by multiplication by i and $-i$ respectively. The space $V^{1,0}$ is spanned by the vectors $e_j = f_j - if_{j+n}$, and the space $V^{0,1}$ is spanned by the vectors $\bar{e}_j = f_j + if_{j+n}$. Let e^1, \dots, e^n and $\bar{e}^1, \dots, \bar{e}^n$ be the corresponding dual base of $(V^{1,0})^*$ and $(V^{0,1})^*$ respectively. Then (9.8) may be rewritten as

$$iF_x^{\mathcal{L}} = \frac{i}{2} \sum_{j=1}^n r_j e^j \wedge \bar{e}^j.$$

We will need the following simple

Lemma 9.8. *Let μ_1, \dots, μ_l and r_1, \dots, r_l be as above. Then $\mu_i = |r_i|$, for any $i = 1, \dots, l$. In particular,*

$$\text{Tr}^+ \tilde{J} = |r_1| + \dots + |r_l|.$$

Proof. Clearly, the vectors $e_1, \dots, e_n; \bar{e}_1, \dots, \bar{e}_n$ form a basis of eigenvectors of \tilde{J} and

$$\begin{aligned} \tilde{J}e_j &= ir_j e_j, & \tilde{J}\bar{e}_j &= -ir_j \bar{e}_j & \text{for } j = 1, \dots, l, \\ \tilde{J}e_j &= \tilde{J}\bar{e}_j = 0 & & & \text{for } j = l+1, \dots, n. \end{aligned}$$

Hence, all the nonzero eigenvalues of \tilde{J} are $\pm i|r_1|, \dots, \pm i|r_l|$. □

9.9. Spinors.

Set

$$S^+ = \bigoplus_{j \text{ even}} \Lambda^j(V^{0,1}), \quad S^- = \bigoplus_{j \text{ odd}} \Lambda^j(V^{0,1}). \quad (9.9)$$

Define a graded action of the Clifford algebra $C(V)$ on the graded space $S = S^+ \oplus S^-$ as follows (cf. Subsection 4.2): if $v \in V$ decomposes as $v = v^{1,0} + v^{0,1}$ with $v^{1,0} \in V^{1,0}$ and $v^{0,1} \in V^{0,1}$, then its Clifford action on $\alpha \in E$ equals

$$c(v)\alpha = \sqrt{2} (v^{0,1} \wedge \alpha - \iota(v^{1,0})\alpha). \quad (9.10)$$

Then (cf. [5, §3.2]) S is the *spinor representation* of $C(V)$, i.e., the complexification $C(V) \otimes \mathbb{C}$ of $C(V)$ is isomorphic to $\text{End}(S)$. In particular, the Clifford module E can be decomposed as

$$E = S \otimes W,$$

where $W = \text{Hom}_{C(V)}(S, E)$. The action of $C(V)$ on E is equal to $a \mapsto c(a) \otimes 1$, where $c(a)$ ($a \in C(V)$) denotes the action of $C(V)$ on S . The natural grading on E is given by $E^\pm = S^\pm \otimes W$.

To prove Proposition 9.5 it suffices now to study the action of $\mathbf{c}(F)$ on S . The latter action is completely described by the following

Lemma 9.10. *The vectors $\bar{e}^{j_1} \wedge \cdots \wedge \bar{e}^{j_m} \in S$ form a basis of eigenvectors of $\mathbf{c}(F)$ and*

$$\mathbf{c}(F) \bar{e}^{j_1} \wedge \cdots \wedge \bar{e}^{j_m} = \left(\sum_{j' \notin \{j_1, \dots, j_m\}} r_{j'} - \sum_{j'' \in \{j_1, \dots, j_m\}} r_{j''} \right) \bar{e}^{j_1} \wedge \cdots \wedge \bar{e}^{j_m}.$$

Proof. The proof is an easy computation which is left to the reader. □

9.11. Proof of Proposition 9.5. Recall that the orientation of V is fixed and that we have chosen the basis f_1, \dots, f_{2n} of V which defines the same orientation. Suppose now that the bilinear form F is non-degenerate. Then $l = n$ in (9.8). It is clear, that the orientation defined by iF coincides with the given orientation of V if and only if the number q of negative numbers among r_1, \dots, r_n is even. Hence, by Lemma 9.10, the restriction of $\mathbf{c}(F) \in \text{End}(S)$ on $\Lambda^j(V^{0,1}) \subset S$ is greater than $-(\tau - 2m)$ if the parity of j and q are different. The Proposition 9.5 follows now from (9.9). □

9.12. Proof of Proposition 7.1. Assume that at least q of the numbers r_1, \dots, r_l are negative and let $m_q > 0$ be the minimal positive number such that at least q of these numbers are not greater than $-m_q$. It follows from Lemma 9.10, that

$$\langle c(F)\alpha, \alpha \rangle \geq -(\tau - 2m_q) \|\alpha\|^2, \quad (9.11)$$

for any $j < q$ and any $\alpha \in \Lambda^j(V^{0,1})$.

Clearly, in the conditions of Proposition 7.1, the operator A defined in (9.7) is Γ -invariant, and, hence, bounded. Proposition 7.1 follows now from (9.11) and the Lichnerowicz formula (9.3). □

9.13. Proof of Proposition 7.4. Let $\pi_i : L^2 \mathcal{A}^{0,*}(M, \mathcal{E} \otimes \mathcal{L}^k) \rightarrow L^2 \mathcal{A}^{0,i}(M, \mathcal{E} \otimes \mathcal{L}^k)$ denote the projection and set

$$\tilde{\nabla}^{\mathcal{E} \otimes \mathcal{L}^k} = \sum_i \pi_i \circ \nabla^{\mathcal{E} \otimes \mathcal{L}^k} \circ \pi_i.$$

Denote $\tilde{\Delta}_k = (\tilde{\nabla}^{\mathcal{E} \otimes \mathcal{L}^k})^* \tilde{\nabla}^{\mathcal{E} \otimes \mathcal{L}^k}$. Clearly, $\tilde{\Delta}_k$ preserves the \mathbb{Z} -grading on $L^2 \mathcal{A}^{0,*}(M, \mathcal{E} \otimes \mathcal{L}^k)$. It follows from the proof of Theorem 2.16 in [11], that there exists a sequence $\varepsilon_1, \varepsilon_2, \dots$, convergent to zero, such that

$$(1 - \varepsilon_k) \langle \tilde{\Delta}_k \gamma, \gamma \rangle - \varepsilon_k k \|\gamma\|^2 \leq \langle \Delta_k \gamma, \gamma \rangle \leq (1 + \varepsilon_k) \langle \tilde{\Delta}_k \gamma, \gamma \rangle + \varepsilon_k k \|\gamma\|^2,$$

for any γ in the domain of Δ_k . Hence,

$$\begin{aligned} \|\langle (\Delta_k - \tilde{\Delta}_k) \gamma, \gamma \rangle\| \\ \leq \varepsilon_k \langle \tilde{\Delta}_k \gamma, \gamma \rangle + \varepsilon_k k \|\gamma\|^2 \leq \varepsilon_k \sum_i \langle \Delta_k \pi_i \gamma, \pi_i \gamma \rangle + \varepsilon_k k \|\gamma\|^2. \end{aligned}$$

Suppose now that $\gamma = \alpha + \beta$, where $\alpha \in \text{Dom}^i(D_k^2)$, $\beta \in \text{Dom}^j(D_k^2)$, $i \neq j$. Then

$$\begin{aligned} \|2 \text{Re} \langle \Delta_k \alpha, \beta \rangle\| &= \|2 \text{Re} \langle (\Delta_k - \tilde{\Delta}_k) \alpha, \beta \rangle\| \\ &\leq \|\langle (\Delta_k - \tilde{\Delta}_k) \gamma, \gamma \rangle\| + \|\langle (\Delta_k - \tilde{\Delta}_k) \alpha, \alpha \rangle\| + \|\langle (\Delta_k - \tilde{\Delta}_k) \beta, \beta \rangle\| \\ &\leq 2\varepsilon_k \langle \Delta_k \alpha, \alpha \rangle + 2\varepsilon_k \langle \Delta_k \beta, \beta \rangle + 2\varepsilon_k k \|\alpha\|^2 + 2\varepsilon_k k \|\beta\|^2. \end{aligned}$$

Similarly one obtains an estimate for the imaginary part of $\langle \Delta_k \alpha, \beta \rangle$. This leads to the following analogue of Proposition 7.4 for the operator Δ_k :

$$\|\langle \Delta_k \alpha, \beta \rangle\| \leq 2\varepsilon_k \langle \Delta_k \alpha, \alpha \rangle + 2\varepsilon_k \langle \Delta_k \beta, \beta \rangle + 2\varepsilon_k k \|\alpha\|^2 + 2\varepsilon_k k \|\beta\|^2. \quad (9.12)$$

We now apply the Lichnerowicz formula (9.6) to obtain Proposition 7.4 from (9.12). Note, first, that the operator $A \in \text{End}(\mathcal{E}) \subset \text{End}(\mathcal{E} \otimes \mathcal{L}^k)$, defined in (9.7), is independent of k and bounded. Note, also, that, by Lemma 9.10, the operator $\mathbf{c}(F^{\mathcal{L}})$ preserves the \mathbb{Z} -grading on $L^2 \mathcal{A}^{0,*}(M, \mathcal{E} \otimes \mathcal{L}^k)$. Hence, it follows from (9.12) and the Lichnerowicz formula (9.6) that

$$\begin{aligned} \|\langle D_k^2 \alpha, \beta \rangle\| &\leq |\langle \Delta_k \alpha, \beta \rangle| + |\langle A \alpha, \beta \rangle| \\ &\leq 2\varepsilon_k \langle \Delta_k \alpha, \alpha \rangle + 2\varepsilon_k \langle \Delta_k \beta, \beta \rangle + 2\varepsilon_k k \|\alpha\|^2 + 2\varepsilon_k k \|\beta\|^2 + \|A\| \|\alpha\| \|\beta\| \\ &\leq 2\varepsilon_k \langle D_k^2 \alpha, \alpha \rangle + 2\varepsilon_k \langle D_k^2 \beta, \beta \rangle + 2\varepsilon_k k (1 + \|\mathbf{c}(F^{\mathcal{L}})\| + 2\|A\|) \|\alpha\|^2 \\ &\quad + 2\varepsilon_k k (1 + \|\mathbf{c}(F^{\mathcal{L}})\| + 2\|A\|) \|\beta\|^2. \end{aligned}$$

Hence, Proposition 7.4 holds with $\delta_k = (1 + \|\mathbf{c}(F^{\mathcal{L}})\| + 2\|A\|) \varepsilon_k$. \square

10. ESTIMATE OF THE METRIC LAPLACIAN

In this section we prove Proposition 6.4. The proof consists of two steps. First, we establish the following

Lemma 10.1. *Suppose that $F^{\mathcal{L}}$ does not vanish at any point $x \in M$. Fix a compact subset $K \subset M$ and a positive number $\varepsilon > 0$. There exists a constant $C_{K,\varepsilon}$ such that, for any $k \in \mathbb{Z}$ and any smooth section $s \in \Gamma(M, \mathcal{E} \otimes \mathcal{L}^k)$ supported on K , one has*

$$\langle \Delta_k s, s \rangle \geq k \langle (\tau(x) - \varepsilon)s, s \rangle - C_{K,\varepsilon} \|s\|^2. \quad (10.1)$$

Then we use the fact Δ_k is Γ -invariant and that M/Γ is compact to deduce Proposition 6.4 from Lemma 10.1. This is done in Subsections 10.5, 10.7.

We now pass to the proof of Lemma 10.1. The proof is almost verbatim repetition of the proof of Proposition 4.4 in [8] (see also [16, §2], [7, §2]) and occupies Subsections 10.2–10.4.

10.2. Reduction to a scalar operator. In this subsection we construct a space \mathcal{Z} and an operator $\tilde{\Delta}$ on the space $L^2(\mathcal{Z})$ of \mathcal{Z} , such that the operator Δ_k is “equivalent” to a restriction of $\tilde{\Delta}$ onto certain subspace of $L^2(\mathcal{Z})$. This allow to compare the operators Δ_k for different values of k .

Let \mathcal{F} be the principal G -bundle with a compact structure group G , associated to the vector bundle $\mathcal{E} \rightarrow M$. Let \mathcal{Z} be the principal $(S^1 \times G)$ -bundle over M , associated to the bundle $\mathcal{E} \otimes \mathcal{L} \rightarrow M$. Then \mathcal{Z} is a principle S^1 -bundle over \mathcal{F} . We denote by $p : \mathcal{Z} \rightarrow \mathcal{F}$ the projection.

The connection $\nabla^{\mathcal{L}}$ on \mathcal{L} induces a connection on the bundle $p : \mathcal{Z} \rightarrow \mathcal{F}$. Hence, any vector $X \in T\mathcal{Z}$ decomposes as a sum

$$X = X^{\text{hor}} + X^{\text{vert}}, \quad (10.2)$$

of its horizontal and vertical components.

Consider the *horizontal exterior derivative* $d^{\text{hor}} : C^\infty(\mathcal{Z}) \rightarrow \mathcal{A}^1(\mathcal{Z}, \mathbb{C})$, defined by the formula

$$d^{\text{hor}} f(X) = df(X^{\text{hor}}), \quad X \in T\mathcal{Z}.$$

The connections on \mathcal{E} and \mathcal{L} , the Riemannian metric on M , and the Hermitian metrics on \mathcal{E}, \mathcal{L} determine a natural Riemannian metrics $g^{\mathcal{F}}$ and $g^{\mathcal{Z}}$ on \mathcal{F} and \mathcal{Z} respectively, cf. [7, Proof of Theorem 2.1]. Let $(d^{\text{hor}})^*$ denote the adjoint of d^{hor} with respect to the scalar products induced by this metric. Let

$$\tilde{\Delta} = (d^{\text{hor}})^* d^{\text{hor}} : C^\infty(\mathcal{Z}) \rightarrow C^\infty(\mathcal{Z})$$

be the *horizontal Laplacian* for the bundle $p : \mathcal{Z} \rightarrow \mathcal{F}$.

Let $C^\infty(\mathcal{Z})_k$ denote the space of smooth functions on \mathcal{Z} , which are homogeneous of degree k with respect to the natural fiberwise circle action on the circle bundle $p : \mathcal{Z} \rightarrow \mathcal{F}$. It is shown in [7, Proof of Theorem 2.1], that to prove Lemma 10.1 it suffices to prove (6.2) for the restriction of $\tilde{\Delta}$ to the space $C^\infty(\mathcal{Z})_k$.

10.3. The symbol of $\tilde{\Delta}$. The decomposition (10.2) defines a splitting of the cotangent bundle $T^*\mathcal{Z}$ to \mathcal{Z} into the horizontal and vertical subbundles. For any $\xi \in T^*\mathcal{Z}$, we denote by ξ^{hor} the horizontal component of ξ . Then, one easily checks (cf. [7, Proof of Theorem 2.1]), that the principal symbol $\sigma_2(\tilde{\Delta})$ of $\tilde{\Delta}$ may be written as

$$\sigma_2(\tilde{\Delta})(z, \xi) = g^{\mathcal{F}}(\xi^{\text{hor}}, \xi^{\text{hor}}). \quad (10.3)$$

The subprincipal symbol of $\tilde{\Delta}$ is equal to zero.

On the *character set* $\mathcal{C} = \{(z, \xi) \in T^*\mathcal{Z} \setminus \{0\} : \xi^{\text{hor}} = 0\}$ the principal symbol $\sigma_2(\tilde{\Delta})$ vanishes to second order. Hence, at any point $(z, \xi) \in \mathcal{C}$, we can define the *Hamiltonian map* $F_{z, \xi}$ of $\sigma_2(\tilde{\Delta})$, cf. [17, §21.5]. It is a skew-symmetric endomorphism of the tangent space $T_{z, \xi}(T^*\mathcal{Z})$. Set

$$\text{Tr}^+ F_{z, \xi} = \nu_1 + \cdots + \nu_l,$$

where $i\nu_1, \dots, i\nu_l$ are the nonzero eigenvalues of $F_{z, \xi}$ for which $\nu_i > 0$.

Let $\rho : \mathcal{Z} \rightarrow M$ denote the projection. Then, cf. [7, Proof of Theorem 2.1]¹,

$$\text{Tr}^+ F_{z, \xi} = \tau(\rho(z)) |\xi^{\text{vert}}| \quad (10.4)$$

Here ξ^{vert} is the vertical component of $\xi \in T^*\mathcal{Z}$, and τ denotes the function defined in (6.1).

10.4. Application of the Melin inequality. Proof of Lemma 10.1. Let D^{vert} denote the generator of the S^1 action on \mathcal{Z} . The symbol of D^{vert} is $\sigma(D^{\text{vert}})(z, \xi) = \xi^{\text{vert}}$. Fix $\varepsilon > 0$, and consider the operator

$$A = \tilde{\Delta} - (\tau(\rho(z)) - \varepsilon) D^{\text{vert}} : C^\infty(\mathcal{Z}) \rightarrow C^\infty(\mathcal{Z}).$$

The principal symbol of A is given by (10.3), and the subprincipal symbol

$$\sigma_1^s(A)(z, \xi) = -(\tau(\rho(z)) - \varepsilon) \xi^{\text{vert}}.$$

It follows from (10.4), that

$$\text{Tr}^+ F_{z, \xi} + \sigma_1^s(A)(z, \xi) \geq \varepsilon |\xi^{\text{vert}}| > 0.$$

Hence, by the Melin inequality ([20], [17, Theorem 22.3.3]), for any compact subset $K \subset \mathcal{Z}$, there exists a constant $C_{K, \varepsilon}$, depending on K and ε , such that

$$\langle Af, f \rangle \geq -C_{K, \varepsilon} \|f\|^2, \quad \text{for any } f \in C^\infty(\mathcal{Z}), \text{ supp } f \subset K. \quad (10.5)$$

Here $\|\cdot\|$ denotes the L^2 norm of the function $f \in C^\infty(\mathcal{Z})$.

From (10.5), we obtain

$$\langle \tilde{\Delta} f, f \rangle \geq \langle (\tau(\rho(z)) - \varepsilon) D^{\text{vert}} f, f \rangle - C_\varepsilon \|f\|^2.$$

Noting that if $f \in C^\infty(\mathcal{Z})_k$, then $D^{\text{vert}} f = kf$, the proof of Lemma 10.1 is complete. \square

We pass to the proof of Proposition 6.4.

10.5. IMS localization formula. If $f \in C^\infty(M)$ and $\gamma \in \Gamma$ we denote by f^γ the function defined by the formula $f^\gamma(x) = f(\gamma^{-1} \cdot x)$. One easily sees (cf. [3, §3]), that there exists a C^∞ function $f : M \rightarrow [0, 1]$ with compact support, such that

$$\sum_{\gamma \in \Gamma} (f^\gamma(x))^2 \equiv 1. \quad (10.6)$$

We need the following version of the IMS localization formula (cf. [10], [9, Lemma 4.10])

¹The absolute value sign of ξ^{vert} is erroneously missing in [7].

Lemma 10.6. *The following identity holds*

$$\Delta_k = \sum_{\gamma \in \Gamma} f^\gamma \Delta_k f^\gamma + \frac{1}{2} \sum_{\gamma \in \Gamma} [f^\gamma, [f^\gamma, \Delta_k]]. \quad (10.7)$$

Proof. Using (10.6), we can write

$$\Delta_k = \sum_{\gamma \in \Gamma} (f^\gamma)^2 \Delta_k = \sum_{\gamma \in \Gamma} \left(f^\gamma \Delta_k f^\gamma + f^\gamma [f^\gamma, \Delta_k] \right).$$

Similarly,

$$\Delta_k = \sum_{\gamma \in \Gamma} \Delta_k (f^\gamma)^2 = \sum_{\gamma \in \Gamma} \left(f^\gamma \Delta_k f^\gamma - [f^\gamma, \Delta_k] f^\gamma \right).$$

Summing these two identities and dividing by 2 we come to (10.7). \square

10.7. Proof of Proposition 6.4. To prove Proposition 6.4 it remains to estimate all the terms in the equality (10.7).

It follows from Lemma 10.1, that for any $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that

$$\langle f \Delta_k f s, s \rangle = \langle \Delta_k f s, f s \rangle \geq k \langle (\tau(x) - \varepsilon) f^2 s, s \rangle - C_\varepsilon f^2 \|s\|^2, \quad s \in \Gamma(M, \mathcal{E} \otimes \mathcal{L}^k).$$

Since the operator Δ_k is Γ -invariant, this inequality remains true if we replace everywhere f with f^γ . Hence, in view of (10.6),

$$\sum_{\gamma \in \Gamma} \langle f^\gamma \Delta_k f^\gamma s, s \rangle \geq k \langle (\tau(x) - \varepsilon) s, s \rangle - C_\varepsilon \|s\|^2. \quad (10.8)$$

We now study the second summand in (10.7). Since Δ_k is a second order differential operator, the double-commutant $[f, [f, \Delta_k]]$ is an operator of multiplication by a function. Let us denote this function by Φ . Then, for any $\gamma \in \Gamma$, the operator $[f^\gamma, [f^\gamma, \Delta_k]]$ acts by multiplication by Φ^γ . Since the support of f is compact so is the support of Φ . It follows that

$$\sum_{\gamma \in \Gamma} [f^\gamma, [f^\gamma, \Delta_k]] = \sum_{\gamma \in \Gamma} \Phi^\gamma \quad (10.9)$$

is a smooth Γ -invariant function. Hence, it is bounded. Set

$$C = \max_{x \in M} \left| \sum_{\gamma \in \Gamma} \Phi^\gamma \right|.$$

From (10.9) we obtain

$$\left\| \sum_{\gamma \in \Gamma} [f^\gamma, [f^\gamma, \Delta_k]] \right\| \leq C. \quad (10.10)$$

Combining (10.7), (10.8) and (10.10) we obtain

$$\langle \Delta_k s, s \rangle = \langle \Delta_k f s, f s \rangle \geq k \langle (\tau(x) - \varepsilon) f^2 s, s \rangle - (C_\varepsilon + \frac{1}{2} C) f^2 \|s\|^2,$$

for any $s \in \Gamma(M, \mathcal{E} \otimes \mathcal{L}^k)$. \square

REFERENCES

- [1] A. Andreotti and H. Grauert, *Théorèmes de finitude pour la cohomologie des espaces complexes*, Bull. Soc. Math. France **90** (1962), 193–259.
- [2] M. Atiyah and W. Schmid, *A geometric construction of the discrete series for semisimple lie groups*, Inventiones Math. **42** (1977), 1–62.
- [3] M. F. Atiyah, *Elliptic operators, discrete groups and von neumann algebras*, Asrérisque **32/33** (1976), 43–72.
- [4] M. Audin, *The topology of torus action on symplectic manifolds*, Progress in Mathematics, vol. 93, Birkhäuser, Basel, Boston, Berlin, 1991.
- [5] N. Berline, E. Getzler, and M. Vergne, *Heat kernels and Dirac operators*, Springer-Verlag, 1992.
- [6] A. Borel, *Compact Clifford-Klein forms on symmetric spaces*, Topology **2** (1963), 11–122.
- [7] D. Borthwick and A. Uribe, *Almost complex structures and geometric quantization*, Math. Res. Lett. **3** (1996), 845–861.
- [8] M. Braverman, *Vanishing theorems for the half-kernel of a Dirac operator*, Preprint, 1998.
- [9] M. Braverman and M. Farber, *Novikov type inequalities for differential forms with non-isolated zeros*, Math. Proc. Cambridge Philos. Soc. **122** (1997), 357–375.
- [10] H.L. Cycon, R.G. Froese, W. Kirsch, and B. Simon, *Schrödinger operators with applications to quantum mechanics and global geometry*, Texts and Monographs in Physics, Springer-Verlag, 1987.
- [11] J.-P. Demailly, *Champs magnétiques et inégalités de Morse pour la d'' -cohomologie*, Ann. Inst. Fourier (Grenoble) **35** (1985), no. 4, 189–229.
- [12] J.-P. Demailly, T. Peternell, and M. Schneider, *Holomorphic line bundles with partially vanishing cohomology*, Proceedings of the Hirzebruch 65 Conference on Algebraic Geometry (Ramat Gan, 1993) (Ramat Gan), Israel Math. Conf. Proc., vol. 9, Bar-Ilan Univ., 1996, pp. 165–198.
- [13] J. J. Duistermaat, *The heat kernel Lefschetz fixed point formula for the spin-c Dirac operator*, Progress in Nonlinear Differential Equations and Their Applications, vol. 18, Birkhäuser, Boston, Basel, Berlin, 1996.
- [14] P. Griffiths, *Hermitian differential geometry and the theory of positive and ample holomorphic vector bundles*, J. Math. Mech. **14** (1965), 117–140.
- [15] P. Griffiths and W. Schmid, *Locally homogeneous complex manifolds*, Acta Math. **123** (1969), 253–301.
- [16] V. Guillemin and A. Uribe, *The Laplacian operator on the n -th tensor power of a line bundle: Eigenvalues which are uniformly bounded in n* , Asymptotic Analysis **1** (1988), 105–113.
- [17] L. Hörmander, *The analysis of linear partial differential operators*, vol. 3, Springer-Verlag, Berlin e.a., 1985.
- [18] R. Langlands, *Dimensions of spaces of automorphic forms*, Proc. of Symposia in Pure Mathematics, vol. IX, Amer. Math. Soc., Providence, R. I., 1966, pp. 253–257.
- [19] H. B. Lawson and M.-L. Michelsohn, *Spin geometry*, Princeton University Press, Princeton, New Jersey, 1989.
- [20] A. Melin, *Lower bounds for pseudo-differential operators*, Ark. Mat. **9** (1971), 117–140.
- [21] M. Vergne, *Geometric quantization and equivariant cohomology*, First European Congress of Mathematics (Paris, 1992), vol. 1, Progr. Math. **119**, Birkhäuser, Boston, 1994, pp. 249–295.

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